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The one-loop functional of chiral SU(2)*

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Abstract: The one-loop functional for chiral SU(2) of Gasser and Leutwyler is extended to include up to three propagators. With this generating functional it is possible to calculate all amplitudes up to order ϕ^6 to next-to-leading order in the low-energy expansion. An external pion is of order ϕ and a vector gauge boson is of order ϕ^2. A check of the existing amplitudes for $\gamma\gamma \to \pi\pi$ is given. The one-loop amplitudes for four-pion production in e^+e^- annihilation are calculated.

Keywords: Spontaneous Symmetry Breaking, Chiral Lagrangians.

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Contents

1. Introduction

In the low-energy region it is not possible to calculate cross sections by means of perturbative QCD because of the high value of the strong coupling constant. One has to use an effective theory. The corresponding theory in this context is called chiral perturbation theory (CHPT) [1]–[4]. It incorporates all symmetries of QCD with the assumption of spontaneous breakdown of chiral symmetry (which is suggested by phenomenological and theoretical evidence [5]), but no other model dependent features. CHPT is nonperturbative in the sense that it is not an expansion in powers of the QCD coupling constant, but an expansion in powers of the external momenta.

For a theory with a spontaneously broken symmetry the Goldstone theorem [6] predicts the existence of massless particles, called Goldstone bosons. In case of CHPT these bosons are not massless because the chiral symmetry is not an exact symmetry, as the quarks have masses. The pions are the by far lightest pseudoscalar mesons and therefore they are identified as the Goldstone bosons of broken chiral SU(2). In CHPT the asymptotic states are not quarks, but pions (or in general pseudoscalar mesons). A spontaneously broken symmetry corresponds to a nonlinear realization a la Nambu-Goldstone [7].

In the calculation of amplitudes one has to include loops to increase the precision and to satisfy unitarity and analyticity [1]. CHPT is not renormalizable in the sense that each loop in the perturbation expansion produces ultraviolet divergences, which have to be renormalized with the help of counterterms of the appropriate order in the external momenta. The finite parts of the coupling constants occurring in the counterterms are not determined by chiral symmetry and must be obtained from experiment or calculated using resonance exchange or other assumptions. Each successive loop and the associated counterterms correspond to successively higher powers of momenta. The higher loops are expected to give only small corrections at appropriately low energies.

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Gasser and Leutwyler have calculated the generating one-loop functional of chiral SU(2) including two propagators in the loop [2]. With this functional one can obtain all amplitudes with at most four external particles, where an external photon counts as two particles. In this article the functional is extended by including a third propagator. This extension makes it easy to analyse processes with up to six external particles. In the whole work equal masses for the two lightest quarks are assumed.

The main purpose of calculations in the framework of pure CHPT is to analyse physics and to investigate QCD in a region ($E \ll 1\,\mathrm{GeV}$) where a perturbative treatment is not possible. CHPT was used successfully to study the low-energy structure of, e.g., $\pi\pi$ scattering [8] or of the pion form factor [9]. There are also important processes with more external particles. One example for an interesting process with six external particles is the scattering $e^+e^- \to 4\pi$, which is analysed in section 4. It is an important cross section for determining the hadronic contribution to the anomalous magnetic moment of the muon and to the fine-structure constant at M_Z.

In section 2 a calculation method with external fields [2] is introduced. The generating functional is given in section 3. The amplitudes of the two channels of $\gamma\gamma \to \pi\pi$ [10, 11, 12] and the one-loop amplitudes of the processes $e^+e^- \to \pi^0\pi^0\pi^+\pi^-$ and $e^+e^- \to 2\pi^+2\pi^-$ are presented in section 4. Finally, section 5 contains some conclusions.

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2. Matrix elements

The lagrangian of massless QCD does not include any terms which connect the right- and left-handed components of the quark fields

$$q_R = \frac{1}{2}(1+\gamma_5)q\,, \qquad q_L = \frac{1}{2}(1-\gamma_5)q\,. \tag{2.1}$$

Thus, the lagrangian is invariant under chiral rotations, i.e. under two independent $\mathrm{U}(N_f)$ transformations (flavour rotations) of the right- and left-handed quark fields.[1] In case of two flavours the Noether currents associated with chiral symmetry are given by:

$$J_I^{\mu 0} = \overline{q}_I \gamma^\mu q_I\,, \qquad J_I^{\mu a} = \overline{q}_I \gamma^\mu \frac{\tau_a}{2} q_I\,, \qquad I = L, R\,. \tag{2.2}$$

The axial and vector currents are defined through the following relations:

$$V^{\mu a} = J_R^{\mu a} + J_L^{\mu a} \qquad \text{vector current,} \tag{2.3}$$

$$A^{\mu a} = J_R^{\mu a} - J_L^{\mu a} \qquad \text{axial current.} \tag{2.4}$$

Because of Lorentz invariance and isospin symmetry there is the following relation:

$$\left\langle 0 | A_a^\lambda(x) | \pi_b(p) \right\rangle = \mathrm{i}\, p^\lambda \delta_{ab} F\, \mathrm{e}^{-\mathrm{i}\, p\cdot x}\,. \tag{2.5}$$

[1] The flavour symmetry of massless QCD should not be mixed up with the still existing colour gauge symmetry.

Since chiral symmetry is spontaneously broken the Goldstone theorem insures that the constant F, which is related to the physical pion decay constant F_π =92.4 GeV through

$$F_\pi = F(1 + \mathcal{O}(m_{\text{quark}}))\,, \tag{2.6}$$

is not zero. As a consequence the axial current, or the divergence of the axial current, plays the role of an interpolating field for the pion.

An external photon couples to the vector current. For example, the matrix element of the scattering $\gamma^* \to 4\pi$ is proportional to the on-shell residue of the corresponding singularity in the Green function

$$\left\langle 0|\mathrm{T}\, V_n^\mu(x_1) A_o^\nu(x_2) A_p^\lambda(x_3) A_q^\rho(x_4) A_r^\sigma(x_5)|0\right\rangle . \tag{2.7}$$

To get such matrix elements one can use the method of external fields [2], which is equivalent to the method of external currents introduced by Schwinger [13]. Extra terms are added to the QCD lagrangian:

$$\mathcal{L}_{QCD} = \overline{q}\,\mathrm{i}\gamma_\mu \nabla^\mu q - \frac{1}{2}\,\mathrm{tr}_c F_{\mu\nu}F^{\mu\nu} + V_{\mu n} v_n^\mu + A_{\mu n} a_n^\mu - S_n s_n + P_n p_n\,. \tag{2.8}$$

∇^μ is the $\mathrm{SU}(3)_C$-covariant derivative. The external fields v_μ, a_μ, and p are set to zero at the end of the calculation. The scalar field s is set equal to the quark mass matrix to give masses to the light quarks, which amounts to explicit symmetry breaking. S_n and P_n are the scalar and pseudoscalar currents:

$$\begin{aligned} S_0 &= \bar{q}q\,, \\ S_i &= \bar{q}\tau_i q \qquad \text{for} \quad i = 1,2,3\,, \\ P_0 &= \mathrm{i}\,\bar{q}\gamma_5 q\,, \\ P_i &= \mathrm{i}\,\bar{q}\gamma_5\tau_i q \qquad \text{for} \ \ i = 1,2,3\,. \end{aligned} \tag{2.9}$$

By adding those terms in eq. (2.8), the global chiral symmetry of the lagrangian is promoted to a local symmetry with the appropriate gauge transformation [2]. The external field method allows the calculation of Green functions like (2.7) in an easy way. The generating functional of the Green functions is of the form:

$$W[v,a,s,p] \doteq \left\langle 0|\mathrm{T}e^{\mathrm{i}\int d^4x\, V_{\mu n}(x)v_n^\mu(x) + A_{\mu n}(x)a_n^\mu(x) - S_n(x)s_n(x) + P_n(x)p_n(x)}|0\right\rangle . \tag{2.10}$$

One gets the Green functions by functional differentiation with respect to external fields, for example:

$$\left.\frac{\delta^3\,\mathrm{i}\,W[v,a,s,p]}{\mathrm{i}^3\delta a_p^\lambda(x_1)\delta a_q^\rho(x_2)\delta v_r^\mu(x_3)}\right|_{\substack{v=0,a=0\\ s=\mathcal{M},p=0}} = \langle 0|\mathrm{T}\, A_{\lambda p}(x_1) A_{\rho q}(x_2) V_{\mu r}(x_3)|0\rangle\,. \tag{2.11}$$

The external fields can be interpreted as fields of external particles:

$$v^\mu + a^\mu = -e\,Q\mathcal{A}^\mu\,, \tag{2.12}$$

$$v^\mu - a^\mu = -e\,Q\mathcal{A}^\mu - \frac{e}{\sqrt{2}\sin\theta_W}(W^{\mu^+}T_+ + \text{h.c.})\,. \tag{2.13}$$

$\mathcal{A}^\mu$ is the photon field, W^{μ^+} the W-boson field, Q the charge matrix and T_+ contains the relevant elements of the Kobayashi-Maskawa matrix.

Because of the large value of the strong coupling constant in the low-energy region it is not possible to calculate the generating functional by means of perturbative QCD. So one has to use an effective lagrangian that includes all possible terms built out of the matrix $\mathrm{U}(\phi)$ (which contains the pion fields) and the external fields that respect all existing symmetries, namely chiral symmetry, Lorentz invariance, charge conjugation and parity. The corresponding lagrangians given by Gasser and Leutwyler [2, 3] are the foundation of CHPT. In case of SU(2) and up to order p^4 (p stands for the external momenta) one has:

$$
\begin{aligned}
\mathcal{L}_2 &= \frac{F^2}{4}\left\langle D_\mu U^\dagger D^\mu U + \chi^\dagger U + \chi U^\dagger \right\rangle , && (2.14)\\
\mathcal{L}_4 &= \frac{1}{4} l_1 \left\langle D_\mu U^\dagger D^\mu U \right\rangle^2 + \frac{1}{4} l_2 \left\langle D_\mu U^\dagger D_\nu U \right\rangle \left\langle D^\mu U^\dagger D^\nu U \right\rangle + \\
&\quad + \frac{1}{16} l_3 \left\langle \chi^\dagger U + \chi U^\dagger \right\rangle^2 + \frac{1}{4} l_4 \left\langle D_\mu U^\dagger D^\mu \chi + D_\mu U D^\mu \chi^\dagger \right\rangle + \\
&\quad + l_5 \left\langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu} \right\rangle + \frac{\mathrm{i}}{4} l_6 \left\langle F^R_{\mu\nu}[D^\mu U, D^\nu U^\dagger] + F^L_{\mu\nu}\left[D^\mu U^\dagger, D^\nu U\right] \right\rangle - \\
&\quad - \frac{1}{16} l_7 \left\langle \chi^\dagger U - \chi U^\dagger \right\rangle^2 + \frac{1}{4} h_1 \left\langle \chi\chi^\dagger + \frac{1}{2}(\chi\tilde{\chi} - \chi^\dagger\tilde{\chi}^\dagger) \right\rangle - \\
&\quad - \left(2h_2 + \frac{l_5}{2}\right) \langle F^R_{\mu\nu} F^{\mu\nu R} + F^L_{\mu\nu} F^{\mu\nu L} \rangle + \frac{1}{4} h_3 \left\langle \chi\chi^\dagger - \frac{1}{2}\left(\chi\tilde{\chi} - \chi^\dagger\tilde{\chi}^\dagger\right) \right\rangle . && (2.15)
\end{aligned}
$$

There are different possible parametrizations of $\mathrm{U}(\phi)$, which all lead to the same on-shell amplitudes. A convenient choice is the exponential parametrization

$$\mathrm{U}(\phi) = \exp\left(\frac{\mathrm{i}\phi}{F}\right), \qquad \phi = \tau_a \phi_a = \begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix}. \tag{2.16}$$

The use of the covariant derivative

$$D_\mu U = \partial_\mu U - \mathrm{i}\, r_\mu\, U + \mathrm{i}\, U\, l_\mu\,, \qquad r_\mu = v_\mu + a_\mu\,, \qquad l_\mu = v_\mu - a_\mu \tag{2.17}$$

is a consequence of adding the extra terms in eq. (2.8). The effective lagrangians (2.14), (2.15) have the same local chiral symmetry as the extended QCD lagrangian in (2.8). χ and $\tilde{\chi}$ are defined as follows:

$$
\begin{aligned}
\chi &= 2B(s + \mathrm{i}\, p)\,, && (2.18)\\
\tilde{\chi} &= \tau_2\, \chi^T \tau_2\,. && (2.19)
\end{aligned}
$$

F, B and the l_i are coupling constants. The Gell-Mann-Oakes-Renner relation, which can also be derived by CHPT, yields:

$$F^2 M^2 = -2\hat{m}\langle 0|\bar{u}u|0\rangle\,, \qquad \text{where} \quad \hat{m} = \frac{(m_u + m_d)}{2}\,. \tag{2.20}$$

At a scale $\mu_{\overline{MS}} = 1\,\mathrm{GeV}$, $\langle 0|\bar{u}u|0\rangle$ is approximately equal to $-(230\,\mathrm{GeV})^3$. Thus, $\hat{m}$ is assumed to be of order M^2:

$$\mathcal{O}(\hat{m}) = \mathcal{O}(M^2) = \mathcal{O}(p^2)\,. \tag{2.21}$$

3. Calculation of the generating functional

CHPT can be defined via the path integral

$$W[v,a,s,p] \doteq \mathcal{N}^{-1} \int d\mu[U]\, \mathrm{e}^{\mathrm{i}\int d^d x\, \mathcal{L}_{\mathrm{eff}}(v,a,s,p)} . \tag{3.1}$$

The calculation is done in d dimensions so that dimensional regularisation can be used.

It can be shown that the loop expansion is an expansion in Planck's constant or in other words an expansion around the solution of the classical equation of motion $\bar{U} = \bar{u}^2$. The matrix U is expanded with the help of a traceless hermitean matrix ξ:

$$U = u^2 = \bar{u}\, \mathrm{e}^{\mathrm{i}\xi}\, \bar{u} . \tag{3.2}$$

To the desired $\mathcal{O}(p^4)$, the loop integration must only be performed over the lagrangian of $\mathcal{O}(p^2)$.

$$W = \mathcal{N}^{-1} \mathrm{e}^{\mathrm{i}\int d^d x \mathcal{L}_4(\bar{U})} \int d\mu[\xi] \mathrm{e}^{\mathrm{i}\int d^d x \mathcal{L}_2(U)} + \cdots . \tag{3.3}$$

$d\mu[\xi]$ is the functional measure for the matrix field ξ. Planck's constant is set equal to 1.

The lagrangian is now expanded in ξ:

$$\begin{aligned} \int d^d x\, \mathcal{L}_2(U) = {} & S_2[\bar{U}] + \frac{1}{4} F^2 \int d^d x \Big\langle D^\mu \left(\bar{u}^\dagger \xi \bar{u}^\dagger\right) D_\mu \left(\bar{u}\xi\bar{u}\right) - \frac{1}{2} D^\mu \bar{U}^\dagger D_\mu \left(\bar{u}\xi^2\bar{u}\right) - \\ & - \frac{1}{2} D^\mu \bar{U} D_\mu \left(\bar{u}^\dagger \xi^2 \bar{u}^\dagger\right) - \frac{1}{2}\xi^2 \left(\bar{u}\chi^\dagger \bar{u} + \bar{u}^\dagger \chi \bar{u}^\dagger\right) \Big\rangle + \\ & + \mathcal{O}\left(\xi^3\right) . \end{aligned} \tag{3.4}$$

One has to consider the terms quadratic in ξ for the one-loop calculation. $S_2[\bar{U}]$ is the classical action to order p^2. With the definitions

$$\begin{aligned} y_\mu &\doteq \frac{1}{2} \bar{u}^\dagger D_\mu \bar{U} \bar{u}^\dagger = -\frac{1}{2} \bar{u} D_\mu \bar{U}^\dagger \bar{u} , && (3.5) \\ \Gamma_\mu &\doteq \frac{1}{2} \left[\bar{u}^\dagger, \partial_\mu \bar{u}\right] - \frac{1}{2} \mathrm{i} \bar{u}^\dagger r_\mu \bar{u} - \frac{1}{2} \mathrm{i} \bar{u} l_\mu \bar{u}^\dagger , && (3.6) \\ \sigma &\doteq \frac{1}{2} \left(\bar{u}\chi^\dagger \bar{u} + \bar{u}^\dagger \chi \bar{u}^\dagger\right) , && (3.7) \\ \hat{\Gamma}^{\mu\, ab} &\doteq -\frac{1}{2} \left\langle \left[\tau^a, \tau^b\right] \Gamma^\mu \right\rangle , && (3.8) \\ \hat{\sigma}^{ab} &\doteq \frac{1}{2} \left\langle \left[\tau^a, y_\mu\right] \left[\tau^b, y^\mu\right] \right\rangle + \frac{1}{4} \left\langle \left\{\tau^a, \tau^b\right\} \sigma \right\rangle , && (3.9) \end{aligned}$$

the generating functional of $\mathcal{O}(p^4)$ has the form

$$W_4 = \mathrm{e}^{\mathrm{i}(S_2[\bar{U}] + S_4[\bar{U}])} \mathcal{N}^{-1} \int d\mu[\xi] \mathrm{e}^{-\mathrm{i}\frac{F^2}{2}\int d^d x \xi^a(x) \mathcal{D}^{ab}(x) \xi^b(x)} . \tag{3.10}$$

The operator $\mathcal{D}$ is

$$\mathcal{D} = \mathbf{1}\Box + \{\hat{\Gamma}^\mu, \partial_\mu\} + \hat{\Gamma}^\mu \hat{\Gamma}_\mu + \hat{\sigma} . \tag{3.11}$$

The factor $\mathcal{N}$ is the one-loop integral with $\mathcal{D}$ replaced by $\mathcal{D}_0$

$$\mathcal{D}_0^{ab} = \delta^{ab}\Box + \frac{1}{2}B\left\langle\left\{\tau^a,\tau^b\right\}\mathcal{M}\right\rangle, \tag{3.12}$$

where $\mathcal{M}$ is the quark mass matrix $\hat{m}\mathbf{1}$. With use of the gaussian formula

$$\int d^d v\, \mathrm{e}^{-\frac{1}{2}v^T A v} = (2\pi)^{\frac{d}{2}}(\det A)^{-\frac{1}{2}} \tag{3.13}$$

one gets

$$W_4 = \mathrm{e}^{\mathrm{i}(S_2+S_4)}\left(\det\frac{\mathcal{D}}{\mathcal{D}_0}\right)^{-\frac{1}{2}}. \tag{3.14}$$

It is well known that one can define a generating functional Z of all *connected* Green functions:

$$Z \doteq -\mathrm{i}\ln W\,, \tag{3.15}$$

$$Z_4 = S_2 + S_4 + \frac{\mathrm{i}}{2}\ln\det\frac{\mathcal{D}}{\mathcal{D}_0}\,. \tag{3.16}$$

With the identity $\ln\det = \mathrm{Tr}\ln$ we have:

$$Z_4 = S_2 + S_4 + \underbrace{\frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\frac{\mathcal{D}}{\mathcal{D}_0}}_{Z_{\text{one-loop}}}\,. \tag{3.17}$$

In the further calculation the operator $\mathcal{D}$ is split into two parts:

$$\mathcal{D} = \mathcal{D}_0 + \delta\,. \tag{3.18}$$

With the quantity $\bar{\sigma}$ defined via

$$\bar{\sigma} \doteq \hat{\sigma} - 2B\hat{m}\mathbf{1}\,, \tag{3.19}$$

δ is of the form

$$\delta = \{\hat{\Gamma}^\mu,\partial_\mu\} + \hat{\Gamma}^\mu\hat{\Gamma}_\mu + \bar{\sigma}\,. \tag{3.20}$$

The one-loop functional can be rewritten as

$$Z_{\text{one-loop}} = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\frac{\mathcal{D}_0+\delta}{\mathcal{D}_0} = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\left(\mathbf{1}+\delta\mathcal{D}_0^{-1}\right) = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln(\mathbf{1}-\delta\Delta)\,. \tag{3.21}$$

The Feynman propagator is the inverse of $-\mathcal{D}_0$. The logarithm is expanded as

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\,. \tag{3.22}$$

Gasser and Leutwyler have calculated the functional with at most two propagators in the loop [2]. In this work I extend it by including a third propagator. The functional is then of the form

$$Z_{\text{one-loop}} = -\frac{1}{2}\,\mathrm{i}\,\mathrm{Tr}(\delta\Delta) - \frac{1}{4}\,\mathrm{i}\,\mathrm{Tr}(\delta\Delta\delta\Delta) - \frac{1}{6}\,\mathrm{i}\,\mathrm{Tr}(\delta\Delta\delta\Delta\delta\Delta) + \mathcal{O}(\phi^8)\,. \tag{3.23}$$

This means the functional is of order ϕ^6 as there are at least two external pions on each vertex in the loop. The photon field can couple to a loop vertex without external pions and is therefore counted as order ϕ^2.

The loop integrals contain the functions A, B which have divergent parts that are proportional to a quantity $\Lambda(\mu)$:

$$
\begin{aligned}
\Lambda(\mu) &= \frac{\mu^{d-4}}{(4\pi)^2}\left(\frac{1}{d-4}-\frac{1}{2}(\ln(4\pi)+\Gamma'(1)+1)\right), && (3.24)\\
A(M^2) &= -\frac{M^2}{(4\pi)^2}\ln\frac{M^2}{\mu^2}-2M^2\Lambda\,, && (3.25)\\
B(p^2,M^2) &= \bar{B}(p^2,M^2)+B(0,M^2)\,, && (3.26)\\
B(0,M^2) &= -\frac{1}{(4\pi)^2}\left(1+\ln\frac{M^2}{\mu^2}\right)-2\Lambda\,. && (3.27)
\end{aligned}
$$

The quantity μ is an arbitrary scale with the dimension of mass. With

$$
\hat{\Gamma}^{\mu\nu} = \partial^\mu\hat{\Gamma}^\nu-\partial^\nu\hat{\Gamma}^\mu+[\hat{\Gamma}^\mu,\hat{\Gamma}^\nu] \qquad (3.28)
$$

the one-loop functional has the following divergence structure [2, 3]:

$$
\begin{aligned}
Z^{\text{div}}_{\text{one-loop}} &= -\frac{\hat{\Lambda}}{12}\,\mathrm{Tr}(\hat{\Gamma}^{\mu\nu}\hat{\Gamma}_{\mu\nu})-\frac{\hat{\Lambda}}{2}\,\mathrm{Tr}(\hat{\sigma}\hat{\sigma})\,, && (3.29)\\
\hat{\Lambda} &= \Lambda(\mu)+\frac{1}{32\pi^2}\ln\frac{M^2}{\mu^2}\,, && (3.30)\\
\mathrm{Tr}(\hat{\Gamma}^{\mu\nu}\hat{\Gamma}_{\mu\nu}) &= -\frac{1}{2}\left\langle D_\mu\bar{U}^\dagger D^\mu\bar{U}\right\rangle^2+\frac{1}{2}\left\langle D_\mu\bar{U}^\dagger D_\nu\bar{U}\right\rangle\left\langle D^\mu\bar{U}^\dagger D^\nu\bar{U}\right\rangle- \\
&\quad -2\left\langle \bar{U}^\dagger F^R_{\mu\nu}\bar{U}F^{L\mu\nu}\right\rangle- \\
&\quad -\mathrm{i}\left\langle F^R_{\mu\nu}[D^\mu\bar{U},D^\nu\bar{U}^\dagger]+F^L_{\mu\nu}[D^\mu\bar{U}^\dagger,D^\nu\bar{U}]\right\rangle- \\
&\quad -\langle F^R_{\mu\nu}F^{\mu\nu R}+F^L_{\mu\nu}F^{\mu\nu L}\rangle\,, && (3.31)\\
\mathrm{Tr}(\hat{\sigma}\hat{\sigma}) &= \frac{1}{4}\left\langle D_\mu\bar{U}^\dagger D^\mu\bar{U}\right\rangle^2+\frac{1}{4}\left\langle D_\mu\bar{U}^\dagger D_\nu\bar{U}\right\rangle\left\langle D^\mu\bar{U}^\dagger D^\nu\bar{U}\right\rangle+ && (3.32)\\
&\quad +\frac{1}{2}\left\langle D_\mu\bar{U}^\dagger D^\mu\bar{U}\right\rangle\left\langle\chi^\dagger\bar{U}+\chi\bar{U}^\dagger\right\rangle+\frac{3}{16}\left\langle\chi^\dagger\bar{U}+\chi\bar{U}^\dagger\right\rangle^2,
\end{aligned}
$$

$$
\begin{aligned}
\left\langle D_\mu\bar{U}^\dagger D^\mu\bar{U}\right\rangle\left\langle\chi^\dagger\bar{U}+\chi\bar{U}^\dagger\right\rangle &= 2\left\langle D_\mu\bar{U}^\dagger D^\mu\chi+D_\mu\bar{U}D^\mu\chi^\dagger\right\rangle+ \\
&\quad +2\left\langle\chi\chi^\dagger+\frac{1}{2}(\chi\tilde{\chi}-\chi^\dagger\tilde{\chi}^\dagger)\right\rangle-\frac{1}{2}\left\langle\chi^\dagger\bar{U}+\chi\bar{U}^\dagger\right\rangle^2. && (3.33)
\end{aligned}
$$

The divergences can be eliminated by a renormalization of the coupling constants:

$$
l_i = \tilde{l}_i+\gamma_i\hat{\Lambda}\,, \qquad h_i = \tilde{h}_i+\delta_i\hat{\Lambda}\,. \qquad (3.34)
$$

The finite quantities $\tilde{l}_i$ and $\tilde{h}_i$ as well as the divergent $\hat{\Lambda}$ are scale independent. To absorb the divergences of the loops, the γ_i and δ_i take the following values [2]:

$$
\begin{aligned}
&\gamma_1 = \frac{1}{3}\,, \quad \gamma_2 = \frac{2}{3}\,, \quad \gamma_3 = -\frac{1}{2}\,, \quad \gamma_4 = 2\,, \quad \gamma_5 = -\frac{1}{6}\,, \\
&\gamma_6 = -\frac{1}{3}\,, \quad \gamma_7 = 0\,, \quad \delta_1 = 2\,, \quad \delta_2 = \frac{1}{12}\,, \quad \delta_3 = 0\,. \qquad (3.35)
\end{aligned}
$$

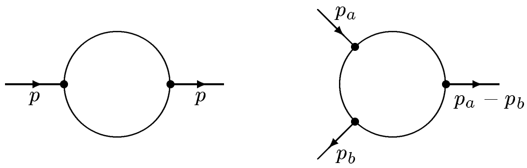

Figure 1: Definition of external momenta.

The renormalization scheme amounts to the following replacements:

$$\begin{aligned} l_i &\to \tilde{l}_i\,, & h_i &\to \tilde{h}_i\,, \\ A &\to 0\,, & B(p^2) &\to \bar{B}(p^2) - \frac{1}{(4\pi)^2}\,. \end{aligned} \tag{3.36}$$

All in all we have the renormalized one-loop functional

$$\begin{aligned} Z_4[U,a,v,s,p] = & \\ & \int d^4x \left(\mathcal{L}_2 + \mathcal{L}_4^{ren}\right) + \int d^4x\, d^4y \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}p(x-y)} \times \\ & \quad \times \Bigg(F_1^{\mu\nu}(p)\, \mathrm{Sp}\left(\hat{\Gamma}^\mu(x)\hat{\Gamma}^\nu(y)\right) + \\ & \qquad + F_2(p)\, \mathrm{Sp}\left(\bar{\sigma}(x)\bar{\sigma}(y)\right) + F_3^{\mu\nu}(p)\, \mathrm{Sp}\left(\hat{\Gamma}^\mu(x)\hat{\Gamma}^\nu(x)\bar{\sigma}(y)\right)\Bigg) + \\ & + \int d^4x\, d^4y\, d^4z \frac{d^4p_a}{(2\pi)^4}\frac{d^4p_b}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p_a(z-x)}\mathrm{e}^{\mathrm{i}p_b(y-z)} \times \\ & \quad \times \Bigg(F_4^{\mu\nu\lambda}(p_a,p_b)\, \mathrm{Sp}\left(\hat{\Gamma}^\mu(x)\hat{\Gamma}^\nu(y)\hat{\Gamma}^\lambda(z)\right) + F_5^{\mu\nu}(p_a,p_b)\, \mathrm{Sp}\left(\hat{\Gamma}^\mu(x)\hat{\Gamma}^\nu(y)\bar{\sigma}(z)\right) \\ & \qquad + F_6^{\mu}(p_a,p_b)\, \mathrm{Sp}\left(\bar{\sigma}(x)\bar{\sigma}(y)\hat{\Gamma}^\mu(z)\right) + F_7(p_a,p_b)\, \mathrm{Sp}\left(\bar{\sigma}(x)\bar{\sigma}(y)\bar{\sigma}(z)\right)\Bigg). \end{aligned} \tag{3.37}$$

$\mathcal{L}_4^{ren}$ is the lagrangian of eq. (2.15) with the l_i and h_i replaced by the $\tilde{l}_i$ and $\tilde{h}_i$. The scale dependence of the renormalized $\mathcal{O}(p^4)$-coupling constants has cancelled the chiral logs of the loops. So everything is scale independent as it should be. The constituent functions $F_1^{\mu\nu}, \ldots, F_7$ can be found in appendix A. Sp denotes the trace in the space of the 3×3 matrices in the adjoint representation. The external momenta flow as shown in figure 1.

The loop part is of order p^4 as one can see by dimensional considerations [1]. With $\mathcal{L}_2$ and $\mathcal{L}_4$ containing all possible terms respecting the symmetries of QCD, Z_4 is the complete generating functional of $\mathcal{O}(p^4)$ for Green functions of quark currents. In the context of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ there is no anomalous part in the generating functional.

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4. Applications

In case of pions and photons one can use the following recipe to extract the scattering amplitudes from the generating functional [14]:

- Expand all quantities which contain the matrices $\bar{U}(\phi)$ or $\bar{u}(\phi)$ in the pion fields.
- Replace the external fields χ by M, a_μ by zero and v_μ by $-e\,Q\mathcal{A}_\mu$ (Q is the quark mass matrix, $\mathcal{A}_\mu$ is the photon field).
- Perform functional differentiation with respect to the external pion and photon fields.

This method is equivalent to the method described in section 2 because the pion matrix is determined by the equation of motion in the following way

$$\phi = (\Box + M^2)^{-1} F\,\partial_\mu a^\mu + \cdots\,, \tag{4.1}$$

and only pole contributions are relevant for scattering amplitudes.

As a check I have recalculated the amplitudes for the processes $\gamma\gamma \to \pi^+\pi^-$ and $\gamma\gamma \to \pi^0\pi^0$. After putting in the expressions for $F_3^{\mu\nu}$ and $F_5^{\mu\nu}$ (that appear in this case) I agree with the results of [10, 11, 12].

An important process with six external particles is the scattering $e^+e^- \to 4\pi$ (four-pion production). The two possible channels are analyed in [15], where the phenomenologically dominant tree and resonance amplitudes for this process are presented. In eq. (4.3) I give the basic part of the one-loop amplitude for the scattering

$$\gamma^*(q) \to \pi^0(p_1)\pi^0(p_2)\pi^+(p_3)\pi^-(p_4) \tag{4.2}$$

from which the complete one-loop amplitude can be derived by use of C invariance and Bose symmetry [15].

$$\begin{aligned}
F_\pi^4 A^\mu_{(4)\mathrm{loop}}(p_1,p_2,p_3,p_4) &= \frac{2\,\left(3\,M_\pi^2 + q^2 - 3\,s - 2\,t_3\right)\,p_{3\nu}\,{F_1}^{\nu\mu}(q^2)}{3\,\left(2\,t_3 - q^2\right)} + \\
&+ \left(\frac{{p_3}^\mu}{2\,t_3 - q^2}\right)\Big\{\left[(p_{1\rho} - p_{3\rho} + q_\rho)\,p_{2\nu} - (p_{1\rho} - p_{3\rho} + q_\rho)\,p_{4\nu}\right]\,{F_1}^{\rho\nu}(q - p_1 - p_3) + \\
&\quad + \left[p_{1\rho}\,(p_{2\nu} - p_{3\nu} + q_\nu) - p_{4\rho}\,(p_{2\nu} - p_{3\nu} + q_\nu)\right]\,{F_1}^{\rho\nu}(p_1 + p_4) + \\
&\quad + 2\Big[\left(3\,M_\pi^4 + 4\,M_\pi^2\,\left(-t_4 + p_3\cdot p_4\right) + 4\,p_1\cdot p_2\,\left(M_\pi^2 - t_4 + p_3\cdot p_4\right)\right)\times \\
&\qquad \times F_2(p_1 + p_2) + 4\,p_1\cdot p_4\,\left(-t_2 + p_2\cdot p_3\right)\,F_2(-p_1 - p_4)\Big]\Big\} \\
&- \frac{p_{3\nu}\,{F_1}^{\mu\nu}(q)}{6} - p_{3\nu}\,{F_1}^{\mu\nu}(p_1 + p_3) - p_{1\nu}\,{F_1}^{\mu\nu}(p_1 + p_4) - \\
&- 4\left[\left(M_\pi^2 + p_1\cdot p_2\right)\,{p_3}^\mu\,F_2(p_1 + p_2) + p_1\cdot p_3\,{p_2}^\mu\,F_2(p_1 + p_3)\right] - \\
&- p_2\cdot p_4\,\left(p_{1\nu} - p_{3\nu}\right)\,{F_3}^{\mu\nu}(q - p_1 - p_3) - \left(M_\pi^2 + 2\,p_1\cdot p_2\right)\,p_{3\nu}\,{F_3}^{\mu\nu}(q - p_3 - p_4) - \\
&- \frac{3\,\mathrm{i}}{4}\,\left(p_{1\nu}\,p_{2\lambda} - p_{3\nu}\,p_{2\lambda} - p_{1\nu}\,p_{4\lambda} + p_{3\nu}\,p_{4\lambda}\right)\,{F_4}^{\mu\nu\lambda}(q, p_1 + p_3) - \\
&- p_2\cdot p_4\,\left(p_{1\nu} - p_{3\nu}\right)\,{F_5}^{\mu\nu}(q, p_1 + p_3) - \left(M_\pi^2 + 2\,p_1\cdot p_2\right)\,p_{3\nu}\,{F_5}^{\mu\nu}(q, p_3 + p_4) - \\
&- \mathrm{i}\,p_1\cdot p_3\,p_2\cdot p_4\,{F_6}^\mu(-p_1 - p_3, p_2 + p_4)\,.
\end{aligned} \tag{4.3}$$

$$s = (p_1 + p_2)^2\,, \qquad t_i = p_i\cdot q \qquad (i = 1,\ldots,4)\,. \tag{4.4}$$

I have dropped terms proportional to q^μ because they cannot contribute to the differential cross section of electroproduction. One can recover them with the help of current conservation. The cross sections for four-pion production and any other details can be found in [15].

5. Conclusions

Chiral perturbation theory is the low-energy effective quantum field theory of QCD. It is characterized by chiral symmetry, by the identification of the lightest mesons as Goldstone bosons associated with the spontaneous breaking of the chiral symmetry and by explicit breaking of chiral symmetry via the masses of the light quarks. To parametrize the breaking terms induced by the quark masses, and also to generate in a systematic way the Green functions of quark currents, it is convenient to insert appropriate sources in the lagrangian that promote the global chiral symmetry to a local symmetry.

One has to include loops to satisfy unitarity and to increase the precision. The loop expansion is an expansion in Planck's constant, i.e an expansion around the solution of the classical equation of motion. The divergences coming from the loops have to be renormalized by counterterms of the corresponding order in the external momenta.

I gave a compact expression for the generating one-loop functional of chiral SU(2) in the isospin limit with at most three propagators in the loop. With this functional one can easily get all possible amplitudes up to order ϕ^6, where an external pion is of order ϕ and a vector gauge boson is of order ϕ^2.

With the help of the generating functional I calculated the processes $e^+e^- \to \pi^0\pi^0\pi^+\pi^-$ and $e^+e^- \to 2\pi^+2\pi^-$ that are relevant for the hadronic contribution to the anomalous magnetic moment of the muon and to the fine-structure constant at M_Z. Apart from that I confirmed the existing results for $\gamma\gamma \to \pi^0\pi^0$ and $\gamma\gamma \to \pi^+\pi^-$ [10, 11, 12]. Other processes that can be calculated with the functional would be, for example, $\gamma^* \to \pi\pi\gamma$ or $\pi\pi \to \pi\pi\pi\pi$.

The corresponding calculation of the generating functional for the symmetry group $\mathrm{SU}(3) \times \mathrm{SU}(3)$ (which allows also K, η to occur as external particles) including the non-leptonic weak interactions [18] is under way.

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A. Constituent functions

The constituent functions of the one-loop functional are of the following form:

$$
\begin{aligned}
F_1^{\mu\nu}(p) &= p^\mu p^\nu\, a(p^2) + g^{\mu\nu}\, b(p^2)\,, \\
F_2(p) &= \frac{1}{4} B(p^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
F_3^{\mu\nu}(p) &= \frac{1}{2} g^{\mu\nu} B(p^2)\,, \\
F_4^{\mu\nu\lambda}(p_a,p_b) &= \mathrm{i}\Big(c_a p_a^\mu p_a^\nu p_a^\lambda + c_b p_b^\mu p_b^\nu p_b^\lambda + d_a p_a^\mu p_a^\nu p_b^\lambda + \\
&\qquad + e_a p_a^\mu p_b^\nu p_a^\lambda + f_a p_b^\mu p_a^\nu p_a^\lambda + d_b p_b^\mu p_b^\nu p_a^\lambda + f_b p_b^\mu p_a^\nu p_b^\lambda + \\
&\qquad + e_b p_a^\mu p_b^\nu p_b^\lambda + g_a p_a^\mu g^{\nu\lambda} + h_a p_a^\nu g^{\mu\lambda} + i_a p_a^\lambda g^{\mu\nu} + \\
&\qquad + h_b p_b^\mu g^{\nu\lambda} + g_b p_b^\nu g^{\mu\lambda} + i_b p_b^\lambda g^{\mu\nu} \Big)\,, \\
F_5^{\mu\nu}(p_a,p_b) &= j_a g^{\mu\nu} + k_a p_a^\mu p_a^\nu + k_b p_b^\mu p_b^\nu + \\
&\quad + l_a p_a^\mu p_b^\nu + m_a p_b^\mu p_a^\nu\,, \\
F_6^\mu(p_a,p_b) &= \mathrm{i}\left(n_a p_a^\mu + n_b p_b^\mu \right), \\
F_7(p_a,p_b) &= \frac{1}{6} C\,,
\end{aligned}
\tag{A.1}
$$

with for example

$$
c_a = c(p_a^2, p_b^2, p_a\cdot p_b) \qquad \text{and} \qquad c_b = c(p_b^2, p_a^2, p_a \cdot p_b)\,, \tag{A.2}
$$

where

$$
\begin{aligned}
a(p^2) &= \frac{B(p^2) - 4\,B_{22}(p^2)}{4}\,, \qquad b(p^2) = -B_{20}(p^2)\,, \\
c_a &= \frac{C_{11\,a} - 4\,C_{22\,a} + 4\,C_{33\,a}}{3}\,, \qquad d_a = \frac{C_{11\,a} - 2\,C_{22\,a} - 2\,\tilde{C}_{22} + 4\,\tilde{C}_{33\,a}}{3}\,, \\
e_a &= \frac{-C/2 + 2\,C_{11\,a} + C_{11\,b} - 2\,C_{22\,a} - 4\,\tilde{C}_{22} + 4\,\tilde{C}_{33\,a}}{3}\,, \\
f_a &= \frac{-2\,\left(\tilde{C}_{22} - 2\,\tilde{C}_{33\,a}\right)}{3}\,, \qquad g_a = \frac{-2\,(C_{20} - 2\,C_{31\,a})}{3}\,, \qquad h_a = \frac{4\,C_{31\,a}}{3}\,, \\
i_a &= \frac{-2\,(C_{20} - 2\,C_{31\,a})}{3}\,, \qquad j_a = -2\,C_{20}\,, \qquad k_a = C_{11\,a} - 2\,C_{22\,a}\,, \\
l_a &= -\frac{C}{2} + C_{11\,a} + C_{11\,b} - 2\,\tilde{C}_{22}\,, \qquad m_a = -2\,\tilde{C}_{22}\,, \qquad n_a = \frac{C}{2} - 2\,C_{11\,a}\,.
\end{aligned}
\tag{A.3}
$$

The B and C functions can be found in App. B.

B. One-loop integrals

The functions B, B_{ij}, C and C_{ij} are defined through the following relations [19, 20, 21] (C_F is the Feynman integration contour):

$$
\{X\} \doteq \frac{1}{\mathrm{i}} \int_{C_F} \frac{d^d k}{(2\pi)^d} \frac{X}{(k^2 - M^2)((k-p)^2 - M^2)}\,, \tag{B.1}
$$

$$
B(p^2) = \{1\}\,, \tag{B.2}
$$

$$
\{k_\mu k_\nu\} = g_{\mu\nu} B_{20}(p^2) + p_\mu p_\nu B_{22}(p^2)\,, \tag{B.3}
$$

$$
\{\{X\}\} \doteq \frac{1}{\mathrm{i}} \int_{C_F} \frac{d^d k}{(2\pi)^d} \frac{X}{(k^2 - M^2)((k-p_a)^2 - M^2)((k-p_b)^2 - M^2)}\,, \tag{B.4}
$$

$$\{\{k_\mu\}\} = p_{a\mu}\, C_{11a} + p_{b\mu}\, C_{11b}\,, \tag{B.5}$$

$$C = \{\{1\}\}\,, \tag{B.6}$$

$$\{\{k_\mu k_\nu\}\} = g_{\mu\nu}\, C_{20} + p_{a\mu}p_{a\nu}\, C_{22a} + p_{b\mu}p_{b\nu}\, C_{22b} + (p_{a\mu}p_{b\nu} + p_{b\mu}p_{a\nu})\, \tilde{C}_{22}\,, \tag{B.7}$$

$$\begin{aligned}\{\{k_\mu k_\nu k_\lambda\}\} = {} & p_{a\mu}p_{a\nu}p_{a\lambda}\, C_{33a} + p_{b\mu}p_{b\nu}p_{b\lambda}\, C_{33b} + \\ & + (p_{a\mu}p_{a\nu}p_{b\lambda} + p_{a\mu}p_{b\nu}p_{a\lambda} + p_{b\mu}p_{a\nu}p_{a\lambda})\, \tilde{C}_{33a} + \\ & + (p_{b\mu}p_{b\nu}p_{a\lambda} + p_{b\mu}p_{a\nu}p_{b\lambda} + p_{a\mu}p_{b\nu}p_{b\lambda})\, \tilde{C}_{33b} + \\ & + (p_{a\mu}g_{\nu\lambda} + p_{a\nu}g_{\mu\lambda} + p_{a\lambda}g_{\mu\nu})\, C_{31a} + \\ & + (p_{b\mu}g_{\nu\lambda} + p_{b\nu}g_{\mu\lambda} + p_{b\lambda}g_{\mu\nu})\, C_{31b}\,, \end{aligned} \tag{B.8}$$

with $C_{ij\,a} = C_{ij}(p_a^2, p_b^2, p_a \cdot p_b)$, $C_{ij\,b} = C_{ij}(p_b^2, p_a^2, p_a \cdot p_b)$ and $C_{ij} = C_{ij\,a} = C_{ij\,b}$. The explicit form of the functions is

$$A(M^2) = M^2 \left(B(0, M^2) + \frac{1}{16\pi^2} \right), \tag{B.9}$$

$$B(p^2, M^2) = \bar{B}(p^2, M^2) + B(0, M^2)\,,$$

$$\bar{B}(p^2, M^2) = \frac{1}{16\pi^2} \left(\sqrt{1 - \frac{4(M^2 - \mathrm{i}\,\epsilon)}{p^2}}\, \ln \frac{\sqrt{1 - \frac{4(M^2 - \mathrm{i}\,\epsilon)}{p^2}} - 1}{\sqrt{1 - \frac{4(M^2 - \mathrm{i}\,\epsilon)}{p^2}} + 1} + 2 \right), \tag{B.10}$$

$$\begin{aligned} C(p_a^2, p_b^2, p_a \cdot p_b, M^2) &= -\frac{1}{16\pi^2\sqrt{\lambda}} \sum_{i=1}^{3} \sum_{\sigma=\pm} \left(\mathrm{Li}_2\left(\frac{x_i}{x_i - z_i^\sigma}\right) - \mathrm{Li}_2\left(\frac{x_i - 1}{x_i - z_i^\sigma}\right) \right), \\ \lambda &= 4\left((p_a \cdot p_b)^2 - p_a^2 p_b^2\right), \qquad p_D^2 = p_a^2 - 2p_a \cdot p_b + p_b^2\,, \\ x_1 &= \frac{1}{2}\left(1 + \frac{p_a^2 - p_b^2 - p_D^2}{\sqrt{\lambda}}\right), \qquad z_1^\sigma = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{p_a^2}(M^2 - \mathrm{i}\,\epsilon)}\right). \end{aligned} \tag{B.11}$$

The quantities x_2, z_2^σ are obtained by the interchange $p_a^2 \leftrightarrow p_D^2$ and x_3, z_3^σ by the interchange $p_a^2 \leftrightarrow p_b^2$. Li_2 denotes the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t} \ln(1 - t)\,. \tag{B.12}$$

The coefficients functions are of the form

$$\begin{aligned} B_{20}(p^2) &= \frac{6\,M^2 - p^2 + 48\,\pi^2\,A + 24\,\pi^2\,\left(4\,M^2 - p^2\right)\,B(p^2)}{288\,\pi^2}\,, \\ B_{22}(p^2) &= \frac{-6\,M^2 + p^2 + 96\,\pi^2\,A - 96\,\pi^2\,\left(M^2 - p^2\right)\,B(p^2)}{288\,p^2\,\pi^2}\,. \end{aligned} \tag{B.13}$$

In the following $B(1)$ is defined as $B(p_b^2)$ and $B(2)$ as $B(p_a^2 - 2\,p_a \cdot p_b + p_b^2)$. The same is valid for B_{20} and B_{22}.

$$C_{11\,a} = \frac{H_{11\,a}\,{p_b}^2 - H_{11\,b}\,p_a \cdot p_b}{{p_a}^2\,{p_b}^2 - (p_a \cdot p_b)^2}\,, \tag{B.14}$$

$$
\begin{aligned}
C_{20} &= \frac{2\,C\,M^2 + \frac{1}{16\,\pi^2} + B(2) - C_{11\,a}\,{p_a}^2 - C_{11\,b}\,{p_b}^2}{4}\,,\\
C_{22\,a} &= \frac{H_{21\,a}\,{p_b}^2 - H_{22\,b}\,p_a\cdot p_b}{{p_a}^2\,{p_b}^2 - (p_a\cdot p_b)^2}\,, \qquad \tilde{C}_{22} = \frac{H_{22\,b}\,{p_a}^2 - H_{21\,a}\,p_a\cdot p_b}{{p_a}^2\,{p_b}^2 - (p_a\cdot p_b)^2}\,,\\
C_{33\,a} &= \frac{H_{31\,a}\,{p_b}^2 - H_{32\,b}\,p_a\cdot p_b}{{p_a}^2\,{p_b}^2 - (p_a\cdot p_b)^2}\,, \qquad \tilde{C}_{33\,a} = \frac{H_{32\,b}\,{p_a}^2 - H_{31\,a}\,p_a\cdot p_b}{{p_a}^2\,{p_b}^2 - (p_a\cdot p_b)^2}\,,\\
C_{31\,a} &= \frac{H_{30\,a}\,{p_b}^2 - H_{30\,b}\,p_a\cdot p_b}{{p_a}^2\,{p_b}^2 - (p_a\cdot p_b)^2}\,,\\
H_{11\,a} &= \frac{-B(1) + B(2) + C\,{p_a}^2}{2}\,, \qquad H_{21\,a} = -C_{20} + \frac{B(2)/2 + C_{11\,a}\,{p_a}^2}{2}\,,\\
H_{22\,a} &= \frac{[-B(1) + B(2)]/2 + C_{11\,b}\,{p_a}^2}{2}\,, \qquad H_{30\,a} = \frac{C_{20}\,{p_a}^2 - B_{20}(1) + B_{20}(2)}{2}\,,\\
H_{31\,a} &= -2\,C_{31\,a} + \frac{{p_a}^2\,C_{22\,a} + B_{22}(2)}{2}\,, \qquad H_{32\,a} = \frac{{p_a}^2\,C_{22\,b} - B_{22}(1) + B_{22}(2)}{2}\,.
\end{aligned}
$$

The divergent functions need to be renormalized as described in section 3.

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